Design of 3D Volumes Using Calculus of Optimization

By: Tony Lyczko

Pierre Massoud

Updated 22 Nov `11

BOMBARDIER

CONFIDENTIAL-

Proprietary Information-For Internal Use Only

Abstract

- The intent of this presentation is to remind the Mass Properties engineer not to overlook the importance that needs to be paid to the external shape and external dimensions of a 3D body in order to ensure that these two elements have been optimized for its intended function.

- A 3D body whose external shape and dimensions have been optimized for its intended use could be considered ``half way`` towards being ``truly`` weight optimized , the other half being the optimization of its internal thicknesses and/or any internal required reinforcement .

- There exists an application of Calculus that enables us to optimize the external shape and dimension of a 3D body subject to the constraints of its intended function. This ensures minimum surface area (S.A.) ,which in turn minimizes the overall weight and cost.

- A brief overview of this application, called ***Lagrange Multipliers***, will be presented along with its application to a few common shapes . It is a useful approach in designing any 3D body where weight is to be minimized.

Design of 3D Volumes Using Calculus of Optimization

- **When Mass Properties engineers refer to optimization of, say, a structural element, we often think about thinning out the internal thickness, scalloping out skins, introducing some lightening holes, etc. all in an effort to try and minimize the internal stress margins; essentially trying to get the margin of safety , MS =0**

- **This is definitely a required step in the weight optimization process, but it should really be considered as a secondary step , ie. "Step 2"**

- **"Step 1" should really be be the optimization of the shape and external dimensions of whatever body we are designing, whether it be a structural body like a beam or a container like a tank.**

Design of 3D Volumes Using Calculus of Optimization

Auxiliary Fuel Tank Case Study Example-

External Belly Tank Vs Internal Fuselage Tank :

- **Auxiliary fuel tanks are often added to aircraft to increase range capability .**
- **These tanks are commonly located inside the fuselage, either above or below the floor . If no room is available for such an installation inside the fuselage, then one common alternative is an external fuselage *belly tank*.**
- **This case study compares the weight and volume capacity of an existing internal fuselage tank to that of a planned external belly tank.**

Estimating Weight

- **With only the external envelope and the required fuel quantity of the proposed belly tank defined , its weight was estimated based on the weight and dimensional data of the existing internal fuse aux tank.**

$$\text{Wt tot} = \underbrace{((\text{SA} \times \text{Thk}) \times \rho}_{\text{Shell weight}} + \text{Internal Struct. Weight})$$

Design of 3D Volumes Using Calculus of Optimization

Real Life Case Study Example:

- **Knowing the actual weight of the existing fuse aux tank and its external shell surface area (SA), thickness and material, its Internal Structure Weight was determined from the above formula.**
- **The ratio of Internal Struct. Weight / Total Tank Weight (Wt tot) turned out to be approximately 50%.**
- **The belly tank weight was then estimated based on its known external SA, an assumed thickness (similar to that of the fuselage tank) ,and an internal structure weight equivalent to 50 % of its total weight**

Design of 3D Volumes Using Calculus of Optimization

Real Life Case Study Example (A “Eureka” Moment):

- **The “Eureka” moment came after we did a side by side comparison of the proposed belly tank with the fuse aux tank, comparing internal volume, SA and weight .**

	Fuse Aux Tank	Belly Tank	Delta Vs Fuse Tank
Surface Area (in2)	11000	12195	+ 11 %
Envelope Volume (in3)	58822	38616	- 34 %
Useable Volume (US Gal)	217	143	- 34 %
Useable Fuel Weight (lb)	1464	965	- 34 %
Weight (lb)	226	251	+11 %

What`s wrong with this picture ?

Design of 3D Volumes Using Calculus of Optimization

Real Life Case Study Example:

The proposed belly tank was to contain 34 % less fuel then the current fuse Aux tank yet was estimated to weigh 11 % more !!!

And

In order to carry about 1000 lb of fuel, the proposed belly tank would have to weigh about 250 lb !!!

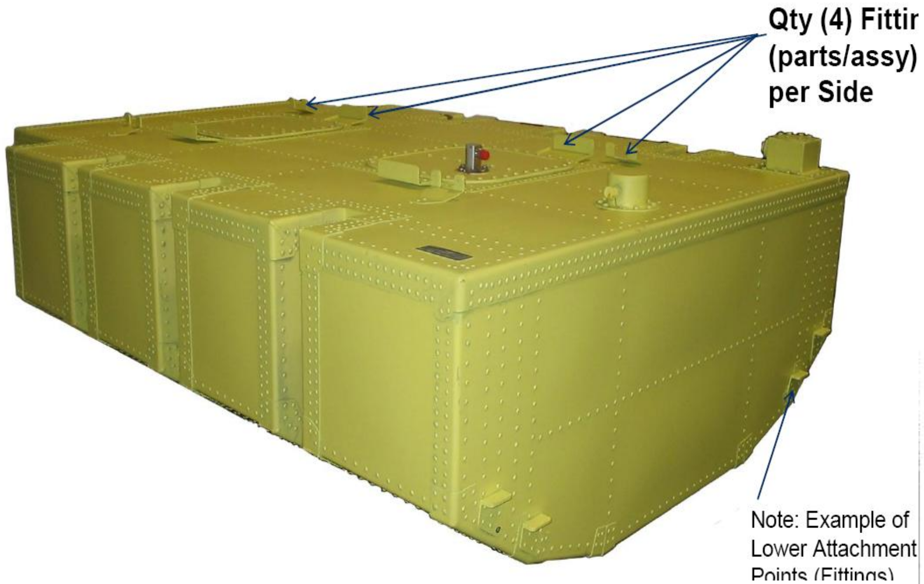
Very Poor Weight (to Useable Volume) Efficiency

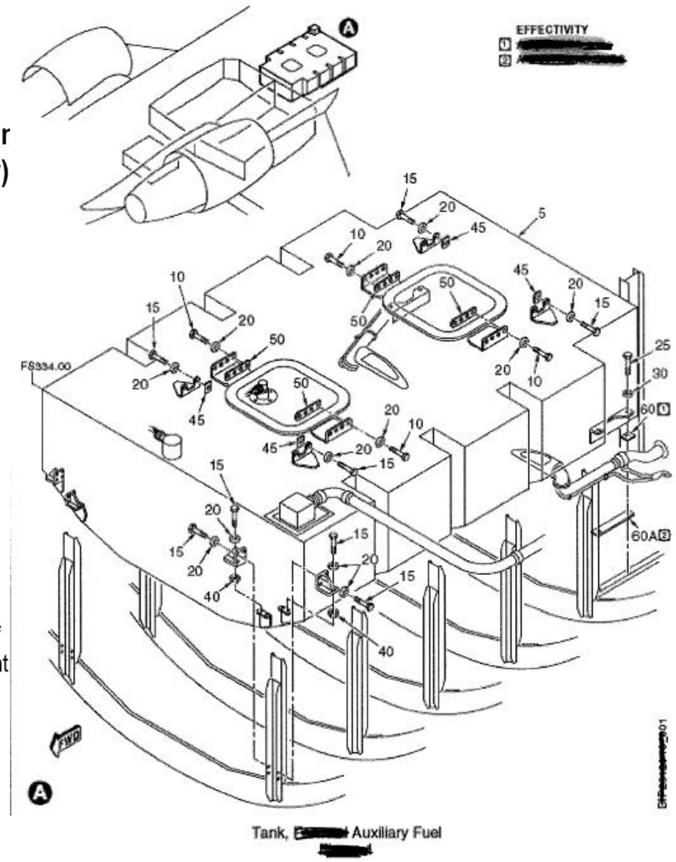
Clearly from the above comparison table, the SA is the weight driver and determines the weight efficiency of the design.

So lets compare the two surface areas and shapes.

Design of 3D Volumes Using Calculus of Optimization

- **Fuse Aux Tank**

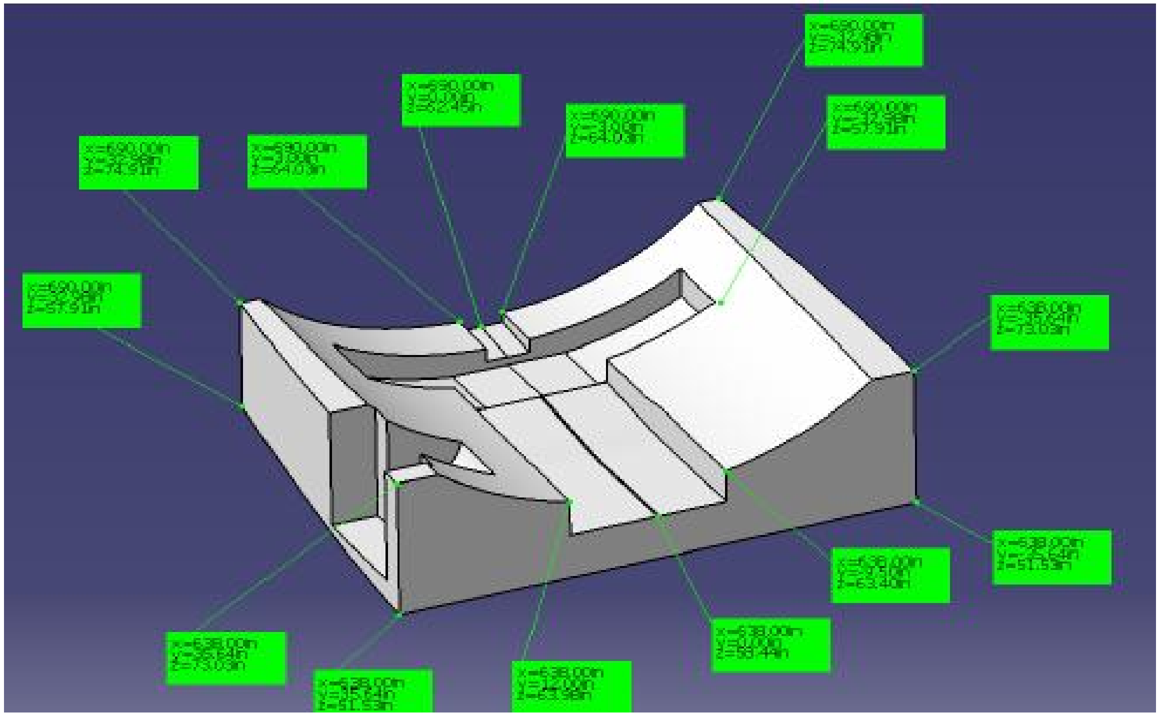




BOMBARDIER

Design of 3D Volumes Using Calculus of Optimization

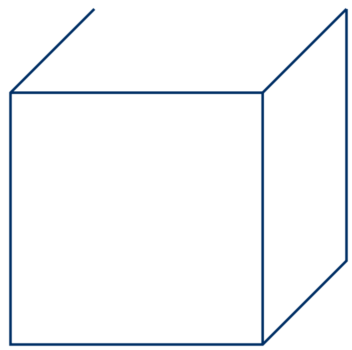
- **<u>Proposed Belly Tank</u>**



- **<u>So clearly a much more complex external shape than the aux fuse tank. Is this the only reason?</u>**

Design of 3D Volumes Using Calculus of Optimization

- **Lets examine this SA and shape effect a little more assuming a common geometric shape, in this case a rectangular box**
- **Lets design a box in order to contain 1000 in3, assuming no dimensional constraints**



- **Assuming each of identical thickness and material.(ignore any internal structure)**

Vol. 1 = 10x15x6.67 = 1000 in3

S.A. 1 = 2(10x15)+2(15x6.67)+2(10x6.67)

= 633.5 in2

Vol.2 = 25x8x5 = 1000 in3

S.A. 2 = 2(8x5)+2(5x25) +2(8x25)

= 730 in2

The weight of Vol. 2 is (730-633.5)/633.5 x100 = 15 % greater than Vol. 1 and therefore one could say *15 % less weight efficient* than Vol.1 in carrying the same volume.

Design of 3D Volumes Using Calculus of Optimization

- **Clearly the outside dimensions drive the magnitude of the Surface Area (SA) , which in turn drives the weight , even though the internal volume of both boxes is identical . The lesson here is that if you want to design a 3D volume (say a tank) at a minimum weight, you have to ask yourself:**

" What is the minimum SA that I should have in order to contain the required volume ?"

- **For those of you who can remember your Calculus course, this is exactly what a certain area of Calculus called "Calculus of Optimization" (or *Lagrange Multipliers*) can solve. This was a"Eureka" moment as it brought back this topic of "Maxima and Minima" problems .**
- **The application of *Lagrange Multipliers* to ,say the SA of a body , enables us to optimize the SA for the given body subject to the the required constraint , eg. Its required volume.**

Design of 3D Volumes Using Calculus of Optimization

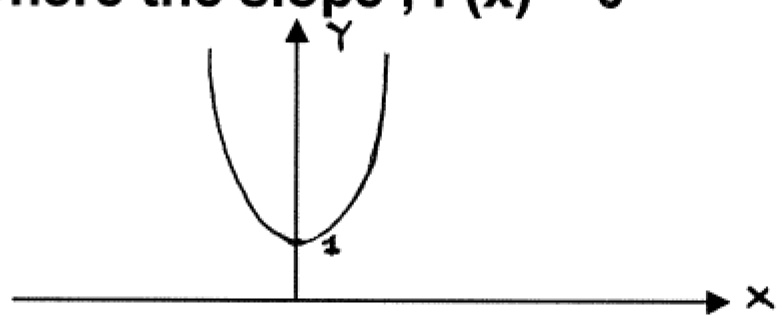
- <u>A Brief overview of Calculus of maxima and minima</u>

Recall for functions of 2 variables, x and y , y= f(x)

The derivative of f(x) = dy/dx = f'(x) is defined as the rate of chang of y with respect to x . Where this rate of change is zero , we ha either a local maxima or a minima of the function f(x) . The rate change is zero where the slope , f'(x) = 0

Example:

Let y = f(x) =X2 +1

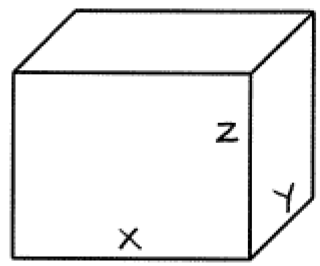


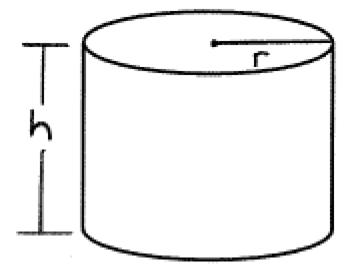
So dy/dx=2x , & setting dy/dx=0 yields a minimum at x=0 , which makes sense

Design of 3D Volumes Using Calculus of Optimization

- <u>Brief overview of Calculus of maxima and minima (cont'd)</u>

In 3D , f(x,y,z) in cartesian coordinates , or f(r,h) in cylindrical coordinates.

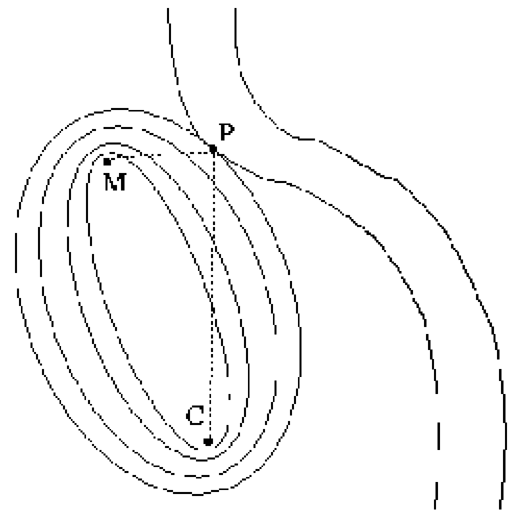




The same technique of maxima and minima can be applied to 3D bodies using "partial" derivatives, ie. $\partial y/\partial x$, $\partial y/\partial z$, $\partial x/\partial z$, or $\partial h/\partial r$, through the application of *Lagrange Multipliers*

Design of 3D Volumes Using Calculus of Optimization

The principle *Lagrange Multipliers*



$f(x_1,x_2 \ldots x_k)$: function to optimize
$g_i(x_1,x_2 \ldots x_k)$: constraint function # i

$$\nabla f = \sum_{1}^{i} \lambda_i \nabla g_i$$

$L(x_1,x_2 \ldots x_k)$: Lagrangian

$$L = f + \sum_{1}^{i} \lambda_i g_i$$

$$\nabla L = 0$$

Number of unknown variables: k + i

$x_1\ x_2 \ldots\ x_k$ & $\lambda_1\ \lambda_2 \ldots\ \lambda_i$

Number of equations: k + i

$$\frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial x_2} = \ldots = \frac{\partial L}{\partial x_k} = 0 \qquad \& \qquad g_1 = g_2 = \ldots = g_i = 0$$

It's a "battle" between the minimizing function 'f' and the constraining function 'g', so in real terms, say a battle to minimize the SA function subject to the constraint of the volume function

Design of 3D Volumes Using Calculus of Optimization

- Lets apply *Lagrange Multipliers* to some common 3D geometric shapes used in aircraft design in order to verify

  1. What is the most optimal shape in terms of SA and Weight?

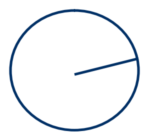
  and

  2. What are the optimal dimensions for the given shape subject to the constraints , eg. required internal volume

Design of 3D Volumes Using Calculus of Optimization

Spherical Volume; Eg. Firex Bottle

Sphere of radius r:



Surface area: $f(r) = 4\pi r^2$

Volume: $g(r) = 4/3\pi r^3$

$L = f + \lambda g = 4\pi r^2 + \lambda(4/3\pi r^3 - V)$

Unknown variables: r, λ

Equations:

$\partial L/\partial r = 8\pi r + \lambda(4\pi r^2) = 0 \qquad \Rightarrow \lambda = -2/r$

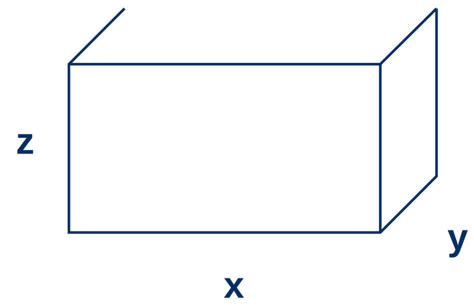
$\partial L/\partial \lambda = 4/3\pi r^3 - V = 0 \qquad \Rightarrow r = \sqrt[3]{3V/4\pi}$

Radius, r, for which SA is Minimum at given volume, V

Design of 3D Volumes Using Calculus of Optimization

Rectangular Volume; Eg. Fuel tank, water tank, avionics box , wing box etc.

Rectangular Box of Dimension x,y,z



Surface area: $f(x, y, z) = 2xy + 2xz + 2yz$

Volume: $g(x, y, z) = xyz$

$$L = f + \lambda g = 2(xy + xz + yz) + \lambda(xyz)$$

Unknown variables: x,y,z, λ

Equations:

$$\partial L / \partial x = 2y + 2z + \lambda yz = 0 \qquad 1. \Rightarrow \lambda = -2(y+z)/yz$$

$$\partial L / \partial y = 2x + 2z + \lambda xz = 0 \qquad 2. \Rightarrow \lambda = -2(x+z)/xz$$

$$\partial L / \partial z = 2x + 2y + \lambda xy = 0 \qquad 3. \Rightarrow \lambda = -2(x+y)/xy$$

$$\partial L / \partial \lambda = xyz - V = 0$$

$$equate 1 \,\&\, 2. \Rightarrow \lambda = -2(y+z)/yz = -2(x+z)/xz \Rightarrow x = z$$

$$equate 2 \,\&\, 3. \Rightarrow \lambda = -2(x+z)/xz = -2(x+y)xy \Rightarrow y = z$$

$$V = xyz = x^3.$$

$$\Rightarrow x = y = z = \sqrt[3]{V}$$

SA is a Minimum for the given rectangular volume , V, when all 3 dimensions x,y,z are equivalent , ie. the volume is a cube

Design of 3D Volumes Using Calculus of Optimization

- **But what if a dimensional constraint exists such that a cube is not possible to install , ie .one side needs to be less than the other two ?**

- **In this case , we can redo the Lagrange analysis with V= xya, where a <x & y.**
- **What falls out is:**

$$. \Rightarrow \lambda = -2(y + a)/ya$$

$$. \Rightarrow \lambda = -2(x + a)/xa$$

$$yields \Rightarrow x = y$$

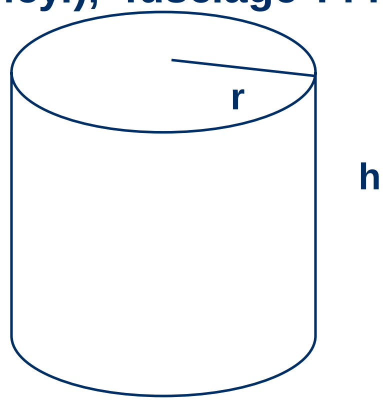
$$V = xya .$$

$$\Rightarrow x = y = \sqrt{V / a}$$

← **If one side has to be = a, then the two remaining sides need to be this**

Design of 3D Volumes Using Calculus of Optimization

Cylindrical Volume; Eg. Water tank, conformal fuel tank , hydraulic accumulator, landing gear strut, actuator cylinder , interior monuments (1/4cyl), fuselage ???, etc.



Cylinder of radius r, height h:

Surface area: $f(r,h) = 2\pi r^2 + 2\pi r h$

Volume: $g(r,h) = \pi r^2 h - V$

$$L = f + \lambda g = 2\pi r^2 + 2\pi r h + \lambda(\pi r^2 h - V)$$

Unknown variables: r, h, λ

Equations:

$$\partial L/\partial h = 2\pi r + \lambda \pi r^2 = 0 \qquad \Rightarrow \lambda = -2/r$$

$$\partial L/\partial r = 4\pi r + 2\pi h + \lambda 2\pi r h = 0 \qquad \Rightarrow h = 2r$$

$$\partial L/\partial \lambda = \pi r^2 h - V = 0 \qquad \Rightarrow r = \sqrt[3]{V/2\pi} \qquad \Rightarrow h = \sqrt[3]{4V/\pi}$$

SA is a Minimum for the given volume , V, when r and h are related to V as shown

Design of 3D Volumes Using Calculus of Optimization

The result of applying Lagrange Multipliers to our previous 1000 in3 volume requirement to 3 defined shapes yields the following :

Volume (in3)	1000

Shape	Side/Radius (in)	Length (in)	Surface (in2)	$Surf^{1/2}/Vol^{1/3}$	Surf/Vol
Cube	10.0		600	2.449	0.6000
Cylinder	5.4	10.8	554	2.353	0.5536
Sphere	6.2		484	2.199	0.4836

-***Recall our previous SA for Vol 1 and Vol 2 was 633.5 in2 and 730 in2, respectively***

-**Comparing 3 potential shapes, it is clear that the spherical shape is the most weight efficient since it has the least SA and SA/Vol ratio, followed by the cylinder and then the cube . (Sphere ⟶ Cyl= +14.5 %, Cyl ⟶ Cube = +8.3%)**

Design of 3D Volumes Using Calculus of Optimization

- **Comparing our proposed Belly Tank to other possible optimized tank configurations**

Volume (in3)	38616

Shape	Side/Radius (in)	Length (in)	Surface (in2)	$Surf^{1/2}/Vol^{1/3}$	Surf/Vol	Weight (lb)*	Delta Wt To Belly Tk (lb)
Belly Tank			12195	3.267	0.3158	251	``
Cube	33.8		6855	2.449	0.1775	141	-110
Rect.Box, a=20 in	43.9	43.9	7377	2.541	0.1910	152	-99
Cylinder	18.3	36.6	6324	2.353	0.1638	130	-121
Sphere	21.0		5525	2.199	0.1431	114	-137

*Wt=(SAxThkxdensity) + Internal Struct wt
where Thk=0.1 in, density=0.103 lb/in3, Internal Struct wt=shell weight

Other Applications – Packaged software : There exist on the market various optimization software which use *Lagrange Multipliers* as a basis of optimization !!!

KUHN-TUCKER CONDITIONS FOR OPTIMALITY

- Kuhn-Tucker conditions for optimality follow directly from a generalization of Lagrange multipliers.
- An optimum design is at hand if:

1. X* is feasible

$$g_j(X^*) \le 0 \qquad j = 1,\dots,m$$
$$h_k(X^*) = 0 \qquad k = 1,\dots,l$$

2. $$\lambda g_j(X^*) = 0 \qquad j = 1,\dots,m$$
$$\lambda_j \ge 0$$

3. $$\underline{\nabla} F(\underline{X}^*) + \sum_{j=1}^{m} \lambda_j \underline{\nabla} g_j(\underline{X}^*) + \sum_{k=1}^{l} \lambda_{k+m} \underline{\nabla} h_k(\underline{X}^*) = 0$$

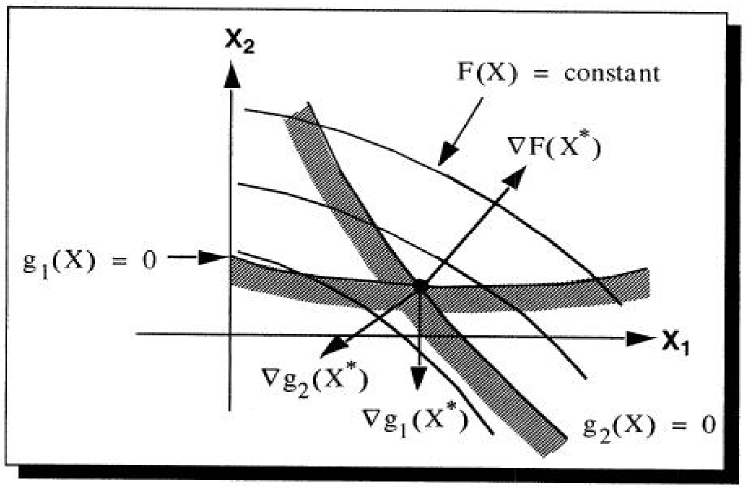
λ_{k+m} unrestricted in sign, but not used in MSC.NASTRAN

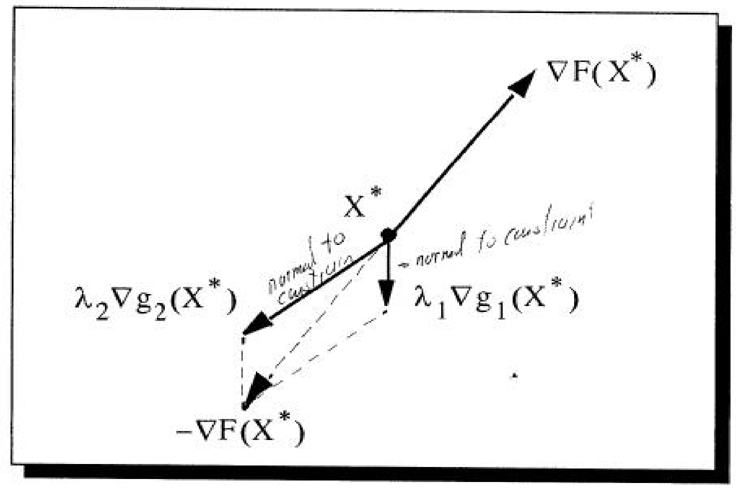
NAS107, Section 2, February 2005

S2-16

MSC SOFTWARE

KUHN-TUCKER CONDITIONS FOR OPTIMALITY (Cont.)



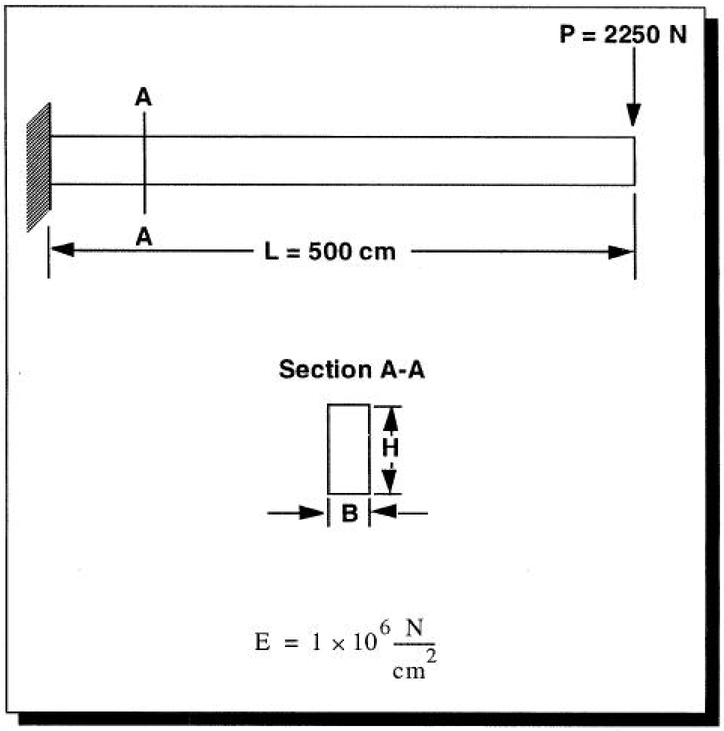


Design of 3D Volumes Using Calculus of Optimization

This Particular example problem will try to optimize the cross sectional dimensions of a simple cantilever beam that is subject to a set of structural constraints using this packaged software

SIMPLE CANTILEVER EXAMPLE

- Problem description



NAS107, Section 2, February 2005

S2-18

MSC SOFTWARE

BOMBARDIER

SIMPLE CANTILEVER EXAMPLE (Cont.)

- Minimize V = B • H • L ← **Minimizing Function , the Volume of the beam , which means its weight**
- Subject to:

$$\delta = \frac{PL^3}{3EI} \leq 2.54 \quad \text{Tip Deflection}$$

$$\sigma = \frac{Mc}{I} \leq 700 \quad \text{Bending Stress}$$

$$\frac{H}{B} \leq 12 \quad \text{Aspect Ratio}$$

$$\left.\begin{array}{l} 1 \leq B \leq 20 \\ 20 \leq H \leq 50 \end{array}\right\} \quad \text{Gauge Requirements}$$

Subject to Structural Constraint Functions

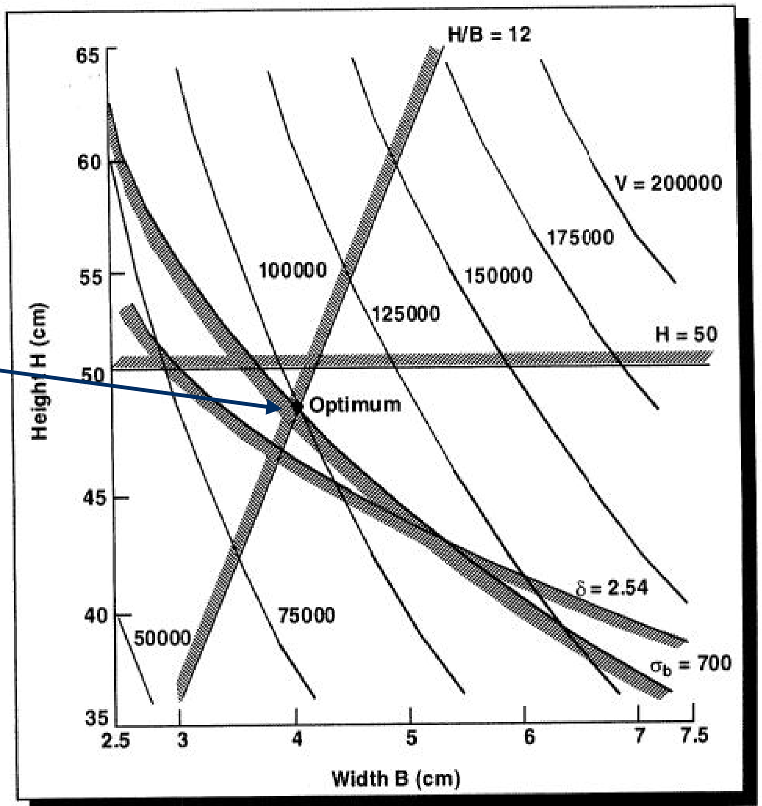
MSC SOFTWARE

BOMBARDIER

SIMPLE CANTILEVER EXAMPLE (Cont.)

- The Design Space

The end result is an optimization plot with the location of a ``sweet spot`` yielding the optimized height, H and width , B, of the beam, subject to the noted constraints



MSC SOFTWARE

BOMBARDIER

SERIES APPROXIMATIONS

- Function gradient information can be used to construct first-order Taylor Series approximations

$$f\left(x^0+\Delta x\right)=f\left(x^0\right)+\left.\frac{df}{dx}\right|_{x^0}\cdot\Delta x+\left.\frac{d^2 f}{dx^2}\right|_{x^0}\cdot\frac{\Delta x^2}{2!}+\ldots$$

$$f\left(x^0+\Delta x\right)=f\left(x^0\right)+\left.\frac{df}{dx}\right|_{x^0}\cdot\Delta x+0\left(\Delta x^2\right)$$

An alternative approach using Taylor Series

- where $0\left(\Delta x^2\right)\cong$ error on the order of Δx^2

SERIES APPROXIMATIONS (Cont.)

- Using the Simple Cantilever to illustrate:
  - Minimize V = B • H • L
  - Design variables B and H
  - Subject to:

$$\sigma = \frac{Mc}{I} = \frac{6PL}{BH^2} \leq 700 \frac{N}{cm^2}$$

$$\delta = \frac{PL^3}{3EI} = \frac{4PL^3}{BH^3E} \leq 2.54\ cm$$

Same problem statement as previous, but with only two constraining functions

SERIES APPROXIMATIONS (Cont.)

- First-order approximations:

$$\tilde{V}\left(B^0+\Delta B, H^0+\Delta H, L\right)=\overline{V}\left(B^0, H^0, L\right)+\left.\frac{\partial V}{\partial B}\right|_{B^0,H^0}\cdot\Delta B+\left.\frac{\partial V}{\partial H}\right|_{B^0,H^0}\cdot\Delta H$$

$$\tilde{\sigma}\left(B^0+\Delta B, H^0+\Delta H, L\right)=\sigma\left(B^0, H^0, L\right)+\left.\frac{\partial \sigma}{\partial B}\right|_{B^0,H^0}\cdot\Delta B+\left.\frac{\partial \sigma}{\partial H}\right|_{B^0,H^0}\cdot\Delta H$$

$$\delta\left(B^0+\Delta B, H^0+\Delta H, L\right)=\delta\left(B^0, H^0, L\right)+\left.\frac{\partial \delta}{\partial B}\right|_{B^0,H^0}\cdot\Delta B+\left.\frac{\partial \delta}{\partial H}\right|_{B^0,H^0}\cdot\Delta H$$

The equations are formulated for the minimizing function and the two constraining functions

- At $\left(B^0, H^0\right)=(6, 45)$

$$V\left(B^0+\Delta B, H^0+\Delta H, L\right)=1.35x10^5+2.25x10^4\Delta B+3.0x10^3\Delta H$$

$$\sigma\left(B^0+\Delta B, H^0+\Delta H, L\right)=555.56-92.593\Delta B-24.691\Delta H$$

$$\delta\left(B^0+\Delta B, H^0+\Delta H, L\right)=2.0576-0.34294\Delta B-0.13717\Delta H$$

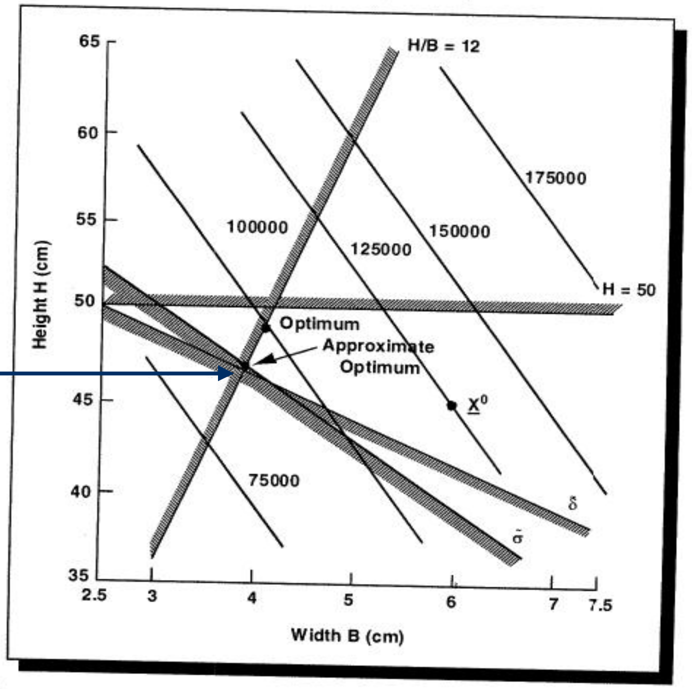
The 3 equations are set up for assuming trial height and width values of 45 and 6, respectively

Design of 3D Volumes Using Calculus of Optimization

SERIES APPROXIMATIONS (Cont.)

- The resultant linearized design space

Generation of an optimization plot with the location of the approximated ``sweet spot`` , which compares quite closely to that previously generated



NAS107, Section 2, February 2005

S2-24

MSC SOFTWARE

BOMBARDIER

Design of 3D Volumes Using Calculus of Optimization

Other Applications of Lagrange Multipliers – Cost Minimization

Example 2-8:

For the process the cost function is:

$$C = 1000P + 4 \times 10^9/PR + 2.5 \times 10^5 R$$

However, C is subject to the inequality constraint equation.

$$PR \leq 9000$$

Adding the slack variable S, as S^2, and forming the Lagrangian function gives:

$$L = 1000P + 4 \times 10^9/PR + 2.5 \times 10^5 R + \lambda(PR + S^2 - 9000)$$

Setting the first partial derivatives of L with respect to P, R, S, and λ equal to zero gives the following four equations:

$$\frac{\partial L}{\partial P} = 1000 - \frac{4 \times 10^9}{P^2 R} + \lambda R = 0$$

$$\frac{\partial L}{\partial R} = 2.5 \times 10^5 - \frac{4 \times 10^9}{PR^2} + \lambda P = 0$$

$$\frac{\partial L}{\partial S} = 2\lambda S = 0$$

$$\frac{\partial L}{\partial \lambda} = PR + S^2 - 9000 = 0$$

Here we have determined a cost function, C, for a given process , which we will try to minimize, and which is subject to a constraining function with variables, P and R.

Design of 3D Volumes Using Calculus of Optimization

Other Applications of Lagrange Multipliers – Cost Minimization

Page

The two cases are $\lambda \neq 0, S = 0$ and $\lambda = 0, S \neq 0$. For the case of $\lambda \neq 0, S = 0$ the equality $PR = 9000$ holds, i.e., the constraint is active. This was the solution obtained in Example 2-6, and the results were:

$C = \$3.44 \times 10^6$ per year $\quad P = 1500$ psi $\quad R = 6 \quad \lambda = -117.3$

For the case of $\lambda = 0, S \neq 0$, the constraint is an inequality, i.e., inactive. This was the solution obtained in Example 2-2 and the results were:

$C = \$3.0 \times 10^6$ per year $\quad P = 1000$psi $\quad R = 4 \quad S = (5000)^{1/2}$

Applying the Lagrange Multipliers yields the minimum cost , C, together with the resulting values, P and R of the constraining function

Design of 3D Volumes Using Calculus of Optimization

- **Lets apply *Lagrange Multipliers* to check out the level of optimization of some popular drink cans**
- **We will obviously have to go back and look at equations used on page 18 when we looked at minimizing the SA of a cylinder.**

The Coke can has a traditional external diameter to length ratio (stubby) whereas the Red Bull can is a long and thin design with a lower external diameter to length ratio. Lets see which one is a more weight (and cost) efficient design, ie. which one better minimizes the SA (and aluminum material) for the required volume of drink and , more importantly , which one comes closest to the Optimized SA for the required Volume.

Design of 3D Volumes Using Calculus of Optimization

- **Volume of a Coke can = 355 ml = 21.663 in3**
- **Radius= 1.30 in , Height = 4.835 in , SA=** $2\pi r^2 + 2\pi rh$
  =50.112 in2
- **Optimized radius for V=21.663 in3 =** $r = \sqrt[3]{V/2\pi}$ **= 1.51 in ,**
- **Optimized height for V=21.663 in3 =** $h = \sqrt[3]{4V/\pi}$ **= 3.02 in**
- **Optimized SA=** $2\pi r^2 + 2\pi rh$
  = 43.0 in2
- **SA (Weight) Efficiency = (1-(50.11-43.0)/43.0))x 100 = 84 % (Not Bad !)**

BOMBARDIER

Design of 3D Volumes Using Calculus of Optimization

- **Volume of a Red Bull can = 250 ml = 15.256 in3**
- **Radius= 1.04 in , Height = 5.320 in , SA=** $2\pi r^2 + 2\pi rh$

  =41.56 in2

- **Optimized radius for V=15.256 in3 =** $r=\sqrt[3]{V/2\pi}$ **= 1.344 in ,**
- **Optimized height for V=15.256 in3 =** $h=\sqrt[3]{4V/\pi}$ **= 2.688 in**
- **Optimized SA=** $2\pi r^2 + 2\pi rh$

  = 34.05 in2

- **SA (Weight) Efficiency = (1-(41.56-34.05)/34.05))x 100 = 78 %**

Conclusion

- **Do not overlook the importance that needs to be paid to the external shape and external dimensions of a 3D body in order to ensure that these two elements have been optimized for its intended function.**
- **The shape of an object is a significant weight driver.**
- **Optimizing the shape and dimensions of an object should be a first step in the weight optimization process, followed by stress optimization**
- ***Lagrange Multipliers* are a powerful Calculus tool that can be used when designing any 3D body where weight is to be minimized.**
- **At least something useful came out of your Calculus course .**

Design of 3D Volumes Using Calculus of Optimization

Thank You for your attention

QUESTIONS ?????

BOMBARDIER